PROCEEDINGS A

rspa.royalsocietypublishing.org

Research

Cite this article: Le Bot A. 2017 Entropy in sound and vibration: towards a new paradigm. *Proc. R. Soc. A* **473**: 20160602. http://dx.doi.org/10.1098/rspa.2016.0602

Received: 28 July 2016
Accepted: 5 December 2016

Subject Areas:
acoustics, mechanics

Keywords:
entropy, sound and vibration, mechanical oscillators, statistical energy analysis

Author for correspondence:
A. Le Bot
e-mail: alain.le-bot@ec-lyon.fr

Entropy in sound and vibration: towards a new paradigm

A. Le Bot

Laboratory of Tribology and System Dynamics CNRS, École centrale de Lyon, Écully, France

ALB, 0000-0002-3834-2685

This paper describes a discussion on the method and the status of a statistical theory of sound and vibration, called statistical energy analysis (SEA). SEA is a simple theory of sound and vibration in elastic structures that applies when the vibrational energy is diffusely distributed. We show that SEA is a thermodynamical theory of sound and vibration, based on a law of exchange of energy analogous to the Clausius principle. We further investigate the notion of entropy in this context and discuss its meaning. We show that entropy is a measure of information lost in the passage from the classical theory of sound and vibration and SEA, its thermodynamical counterpart.

1. Introduction

The idea to use statistical physics methods in acoustics goes back to the work of Sabine [1] at the end of the nineteenth century. In architectural acoustics, the geometrical complexity of rooms and their large size poses major difficulties, and the solving of the wave equation is generally not possible. However, complexity may rather be considered as an advantage if a statistical approach is adopted. In a large auditorium, the sound is rapidly disordered by successive reflections on walls and reaches a state of diffuse field. This results in a particularly simple law, the so-called Sabine's Law, for the decrease of sound in large rooms after extinction of sources.

In the 1960s, the statistical theory of sound had been considerably enlarged by a discovery by Lyon and co-workers [2–5]. They found that sound and vibrational energy flows from high energetic to low energetic regions exactly as heat does in solids. This result is the foundation of statistical energy analysis, a statistical theory of sound and vibration. The equations of statistical

THE ROYAL SOCIETY PUBLISHING

energy analysis are based on an energy balance in each subsystem. The method is quite similar to the application of the first principle of thermodynamics. This allows a direct prediction of the vibrational energy of each subsystem provided that the vibrations are sufficiently disordered and have reached a state of diffuse field. This is the most restrictive assumption in statistical energy analysis.

The question of the second principle of thermodynamics in statistical energy analysis has been examined more recently [6–10]. The main idea is that, because under certain conditions, vibrational energy may be assimilated to heat, it becomes possible to apply Clausius definition of entropy and therefore to introduce a vibrational entropy as well as a vibrational temperature. Several examples of exchange of entropy in mechanical oscillators or continuous systems have been studied in the literature [11,12]. Some of them even show an apparent irreversible behaviour [13,14].

This article is a discussion on the status of entropy in statistical energy analysis with a particular emphasis on its meaning. The paper is organized as follows. Section 2 presents the coupling power proportionality and the analogy with the Clausius principle. Section 3 introduces the concept of entropy in statistical energy analysis, whereas its meaning in terms of information is discussed in §4. In §5, this question is further discussed but in the eyes of the reference theory in which statistical energy analysis is derived. Some concluding remarks are given in §6.

2. Lyon's law and the Clausius principle

The fundamental law found by Lyon and co-workers [2–5] is related to energy flow in sets of mechanical linear oscillators. We shall call it coupling power proportionality. This law states that under certain conditions, the mechanical energy flows from 'hot' to 'cold' regions. This remarkable law suggests an analogy with Clausius principle in thermodynamics. This analogy is the foundation of a statistical theory of 'disordered' sound and vibration called statistical energy analysis. Before to discuss the status of this analogy, let us first introduce Lyon's result.

The most elementary form of the coupling power proportionality examines the power exchanged by two coupled oscillators. Let's define a set of mechanical resonators as shown in figure 1*a*. Each oscillator consists of a moving mass m_i, a spring k_i and a dashpot c_i. The resonators are coupled through springs of stiffness K but not dashpot. The couplings are therefore conservative. The time-varying forces $f_i(t)$ applied to masses are assumed to be stationary random processes. Because the external forces are random, the kinetic and elastic energies of oscillators are also random functions of time. We shall denote by brackets $\langle X\rangle$ the random expectation of a quantity X, and we shall call it mean value of X for short. When the external forces are ergodic, the mean value $\langle X\rangle$ is also the time average of X. We set the following list of assumptions:

— all couplings are conservative,
— all couplings are weak $\epsilon = K/\sqrt{k_i k_j} << 1$ and
— the forces are mutually uncorrelated white noises.

The last assumption means that the cross-correlation function of the forces f_i and f_j is $R_{ij}(\tau) = \langle f_i(t) f_j(t+\tau)\rangle = S_i \delta_{ij} \delta(\tau)$, where δ_{ij} is Kronecker's symbol, $\delta(\tau)$ is the Dirac function and τ is the time delay. Equivalently, the cross-power spectral density is $S_{ij}(\omega) = S_i \delta_{ij}$ and ω is the circular frequency. Under the above conditions, it may be established, by applying Cauchy's theorem in complex analysis, that the mean kinetic and elastic energies are equal for each oscillator (see [15] for a complete proof). Furthermore, it may be proved that the mean vibrational power $\langle P_{12}\rangle$, flowing from resonators 1 to 2 (the choice of these indices is just a convention), is proportional to the difference of the mean vibrational energies $\langle E_1\rangle$, $\langle E_2\rangle$.

$$\langle P_{12}\rangle = \beta(\langle E_1\rangle - \langle E_2\rangle) + o(\epsilon^2), \tag{2.1}$$

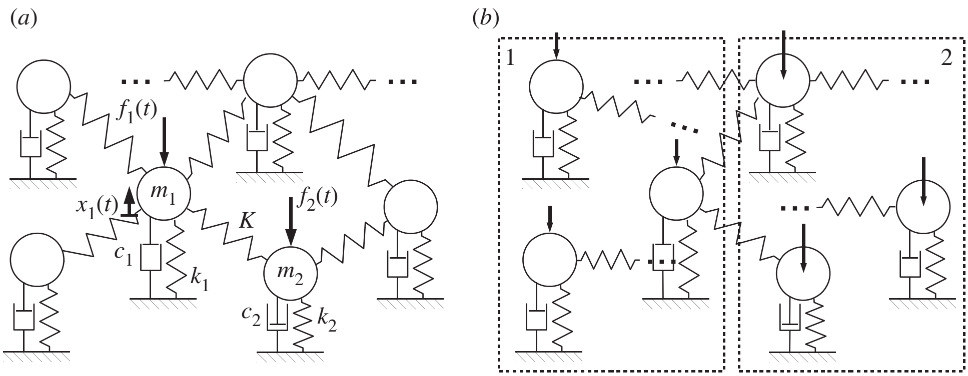

Figure 1. (*a*) Mechanical resonators weakly coupled and loaded by uncorrelated random forces $f_i(t)$. (*b*) Energy exchange between groups of oscillators excited by rain-on-the-roof forces.

where

$$\beta = \frac{K^2(\Delta_1 + \Delta_2)}{m_1 m_2[(\Omega_1^2 - \Omega_2^2)^2 + (\Delta_1 + \Delta_2)(\Delta_1 \Omega_2^2 + \Delta_2 \Omega_1^2)]}. \tag{2.2}$$

In equation (2.2), $\Omega_i = \sqrt{(K + k_i)/m_i}$ are the 'blocked' natural frequencies of resonators and $\Delta_i = c_i/m_i$ their half-power bandwidths. We observe that the proportionality factor β is a function of mechanical parameters of adjacent oscillators only although they belong to a larger set of oscillators. The coupling power proportionality is valid up to order two in the small parameter ϵ. However, in the special case of a set containing only two oscillators, the result is exact.

This result can be generalized by considering groups of oscillators in interaction. The situation is shown in figure 1*b*. We constitute groups of N_i resonators, and we examine the net power exchanged by two such groups (also called subsystems). In each subsystem, resonators are uncoupled. They have different natural frequencies but the same mass m_i and damping coefficient c_i. In addition to the first three assumptions, we set the further assumptions:

— the power spectral density is identical for all resonators of a subsystem (rain-on-the-roof forces),
— the number of resonators is large in each subsystem $N_i >> 1$ and
— the internal damping is light.

Then, the coupling power proportionality takes the form

$$\langle P_{12} \rangle = \beta \left(\frac{\langle E_1 \rangle}{N_1} - \frac{\langle E_2 \rangle}{N_2} \right) + o(\epsilon^2). \tag{2.3}$$

The mechanical power flowing from subsystem 1 to subsystem 2 is proportional to the difference of modal energies $\langle E_i \rangle / N_i$, where $\langle E_i \rangle$ is the mean vibrational energy of subsystem i. The result is again valid up to order two in ϵ. The factor β now depends on all characteristics of the oscillators of the two subsystems in interaction.

The coupling power proportionality is not a result restricted to the study of discrete mechanical resonators. The generalization to flexible structures is straightforward. We know that the dynamics of a flexible structure can always be reduced to that of the eigenmodes. The modes are uncoupled and behave like resonators whose modal mass may be chosen arbitrarily. If the dissipation force acting on the flexible structure is of viscous type, then the half-power bandwidth of modes is also constant. The energy exchange between two flexible structures then reduces to the previous canonical problem of groups of uncoupled oscillators (normal modes form an orthogonal basis). Of course, some difficulties appear in this process. For instance, the analysis of energy exchange is confined to a frequency band $\Delta\omega$ centred on the circular ω. The number of

rspa.royalsocietypublishing.org Proc. R. Soc. A 473: 20160602

modes within $\Delta\omega$ is $N_i = n_i \Delta\omega$, where n_i is the modal density (per rad s^{-1}). However, the main result is maintained (see [15] or [16] for details). Neglecting second-order terms, the Lyon law or coupling power proportionality is

$$P_{12} = \omega\eta_{12} N_1 \left(\frac{E_1}{N_1} - \frac{E_2}{N_2} \right), \tag{2.4}$$

where, from now on, we shall remove brackets for the sake of simplicity. The coupling loss factors, η_{ij}, are functions of the mechanical characteristics of adjacent subsystems. There exact form is of no importance for the following discussion. However, a statistical estimation for two substructures of masses M_1 and M_2 coupled through a weak spring of stiffness K gives the following result:

$$\eta_{12} = \frac{\pi K^2 n_2}{2\omega^3 M_1 M_2}. \tag{2.5}$$

The coupling loss factors verify the reciprocity relationship $\eta_{12} N_1 = \eta_{21} N_2$.

Lyon's Law states that when flexible structures are excited by broadband random forces, the vibrational energy always flows from subsystems having a large modal energy to that with a lower modal energy. The Clausius principle states that when two bodies with different temperatures come into contact, heat flows from the hotter to the colder bodies. The analogy between Lyon's law and the Clausius principle is clear. In sound and vibration, the modal energy plays the role of temperature, so that it is meaningful to claims that even in sound and vibration the vibrational energy flows from 'hot' to 'cold' regions. Of course, there are several constraints for this result to be valid (the above list of assumptions). A careful study of these assumptions leads to establishing that all subsystems must be in diffuse field state (homogeneous and isotropic repartition of energy, see [17]). This state of diffuse field may be interpreted as an equilibrium state with a constant temperature. The analogy is complete. In Lyon's Law, we put in interaction two mechanical structures in vibrational equilibrium. If the coupling is weak, the energy flows from 'hot' to 'cold' structure. In Clausius principle, we put into contact two bodies in thermal equilibrium, then heat flows from hot to cold bodies. Thus, in both principles, the equilibrium is reached in each body (diffuse energy density for vibrating structures, uniform temperature for hot solids) and an intensive quantity (modal energy or temperature) fixes the direction of power flow (vibrational power or heat).

The coupling power proportionality is a powerful tool to construct a statistical theory of sound and vibration. In statistical energy analysis, a complex structure is divided into simple mechanical components (beam, shell, acoustical cavities...) which are weakly coupled. A power balance for each subsystem reads $P_i = P_{\text{diss}} + \sum P_{ij}$, where P_i is the mean power supplied by external forces, P_{diss} the mean power dissipated by viscous forces and P_{ij} the power exchanged with other subsystems. If the state of diffuse field is reached in all subsystems, the coupling power proportionality (2.4) applies, and the power balance becomes

$$P_i = \omega \left[\eta_i E_i + \sum_{j \neq i} \eta_{ij} E_i - \eta_{ji} E_j \right], \tag{2.6}$$

where η_i is a damping loss factor. Equation (2.6) is the foundation of statistical energy analysis. This gives a set of linear equations on the mean energies E_i. The resolution of equation (2.6) leads to the calculation of the mean vibrational energies in all components of the system provided that the injected powers and the coupling loss factors are known.

3. Entropy in statistical energy analysis

The analogy of Lyon's law with the Clausius principle highlights the importance of the two extensive quantities E and N. From the coupling power proportionality and energy balance, it becomes possible to predict the energy levels in all subsystems without solving the wave equation in acoustics or other governing equations for structural components. One must say that all information available on a subsystem reduces to E and N, that is sufficient to provide a closed set of equations.

This is obviously a statistical theory that neglects the details. In particular, statistical energy analysis does not provide the information on the exact repartition of energy among modes at any time but just its mean value per mode. As in statistical physics, we arrive at the notions of macrostate and microstate. The macrostate is characterized by E and N, whereas a microstate is specified by the exact repartition of energy on modes. The most important task is to assess the number of microstates that correspond to a single macrostate.

This problem is a classical question in statistical physics. If we have Z energy quanta and N sites, the number of possibilities to arrange Z among N is

$$W = \frac{(Z+N-1)!}{Z!(N-1)!}. \tag{3.1}$$

Note that in this estimation the energy is undistinguishable, whereas the N sites are distinguishable. Before to derive an explicit relationship of entropy, let us estimate the two main numbers N and Z that appear in the previous expression in the context of acoustics.

In a room, the density of acoustical modes is $n(\omega) = V\omega^2/2\pi^2 c_0^3$, where $c_0 = 340\,\mathrm{m\,s^{-1}}$ is the sound speed. The number of modes whose frequency lie in an octave band $\Delta\omega = \omega/\sqrt{2}$ is $N = n(\omega)\Delta\omega$. For instance, a small room of volume $V = 45\,\mathrm{m^3}$ contains $N \approx 10\,000$ modes in the octave band 1 kHz. This number is of course larger for larger rooms or at higher frequencies. However, usually, the number of modes N in acoustics is of the order of few thousands up to several millions. This is very small compared with the usual number of atoms in statistical physics.

The acoustical energy may be assessed as follows. Consider the same room in which lies a noise of, say, 70 dB that is a root mean squared acoustical pressure of $p = 2\,10^{-5} \times 10^{70/20} \approx 60\,\mathrm{mPa}$. The acoustical energy contained in the whole room is $E = p^2 V/\rho_0 c_0^2$, where $\rho_0 = 1.3\,\mathrm{kg\,m^{-3}}$ is the air density. One obtains $E \approx 1\,\mu\mathrm{J}$. At 1 kHz, the number of energy quanta is therefore $Z = E/\hbar\omega \approx 3.10^{24}$ units where $\hbar$ is Planck's constant.

It appears from this estimation that for ordinary situations in acoustics, the number of energy levels Z is considerably larger than the number of modes N. The sum $Z + N - 1$ differs from Z by a negligible term. Because the product $(Z + N - 1)!/Z!$ contains exactly $N - 1$ terms approximately equal to Z, we may write

$$W = \frac{Z^{N-1}}{(N-1)!}. \tag{3.2}$$

Let us now introduce the Boltzmann's entropy of a system whose number of microstates is W

$$S = k_B \log W, \tag{3.3}$$

where k_B is the Boltzmann's constant. By substituting equation (3.2) into equation (3.3) and using Stirling's approximation $\log(N-1)! = (N-1)\log(N-1) - (N-1)$, one obtains

$$S = k_B(N-1)[\log Z - \log(N-1) + 1]. \tag{3.4}$$

Now, writing N instead of $N - 1$ and $Z = E/\hbar\omega$, we obtain [8]

$$S(E, N) = k_B N \left[1 + \log\left(\frac{E}{\hbar\omega N}\right)\right]. \tag{3.5}$$

This is the final expression of entropy of a vibroacoustical system having energy E spread on N modes about the circular frequency ω. Two fundamental physical constants appear in this relationship, the Planck constant $\hbar$ and the Boltzmann constant k_B.

4. Interpretation in terms of information

We have seen that Lyon's law allows us to construct a thermal theory of sound and vibration in which each subsystem is characterized by two and only two quantities which are the vibrational energy E and the number of modes N. In statistical energy analysis, the state of a subsystem reduces to the knowledge of E and N, and no further information can be provided by the theory.

In the previous section, the entropy was introduced in statistical energy analysis in the same way as in statistical physics for Einstein's solid. In a solid, the energy is localized in atoms. Each atom behaves like a resonator whose discrete energy levels are $E_k = (k + 1/2)\hbar\omega$, $k = 0, 1, \ldots$ by the rules of quantum mechanics. At the macroscopic scale, the state of the solid is specified by its total energy E and the number of atoms N. We may imagine a thought experiment which consists in measuring the exact energy of all resonators at a given time. This exact repartition of energy E on resonators defines the microstate. The gain of information acquired during this process is $I = -\log p$, where p is the probability of the microstate. Because all microstates are assumed to be equiprobable $p = 1/W$, where W is the total number of microstates. The gain of information is therefore $I = \log W$. It becomes clear that Bolzmann's entropy $S = k_B \log W$ is a measure of information lost when we ignore the actual microstate of the solid.

In statistical energy analysis, the vibrational energy is not localized on atoms but on normal modes. The non-local character of modes does not raise any difficulty because they remain well identified by their natural frequencies and mode shapes. Modes are therefore discernable like atoms. Following the analogy with Einstein's solid, the entropy given in equation (3.5) is a measure of information lost when we ignore the actual repartition of energy on modes. This is exactly what happens in statistical energy analysis, and we can conclude that equation (3.5) gives the information lost in statistical energy analysis.

In this reasoning, we have tacitly assumed that a microscopic configuration is fully determined by the knowledge of the exact repartition of energy on modes. This raises the question of the definition of a microstate in classical physics. In acoustics, the vibrational field is the solution to the wave equation. This solution can always be developed in a series of normal modes. At any time, a mode has a kinetic energy that may be different from its elastic energy. This information on the repartition of the kinetic and elastic energies has not be taken into account in equation (3.5). Another point of view is to start from geometrical acoustics. Here, we consider geometrical acoustics in its strict sense that is when phase of rays has been neglected. The sound field may then be viewed as a sound particle gaze of speed c_0. In this context, a microstate is rather defined by the specific energy density $\rho(\mathbf{x}, \mathbf{v})$ in the phase space $(\mathbf{x}, \mathbf{v})$, where $\mathbf{x}$ is the position and $\mathbf{v}$ the velocity of sound particles.

The problem of interpretation of equation (3.5) raises the question of the theoretical framework in which statistical energy analysis is interpreted as a derived theory. This is discussed in §5.

5. Towards a new paradigm

Statistical energy analysis occupies a special place in physical sciences. Because several centuries, the general trend in physics is to construct more general theories whose range of application is enlarged. For instance, magnetism and electrostatics have been replaced by Maxwell electromagnetism which, in turn, has been replaced by Feynman quantum electrodynamics. In this process, the older theories remain operative but only within a certain domain of validity which may be delimited in the upper theory.

In acoustics, the oldest theory is certainly geometrical acoustics yet known by Ancient Greeks. The wave theory of sound initiated in the eighteenth century by D'Alembert generalizes geometrical acoustics while allowing to explain other phenomena such as diffraction, interferences and the existence of modes. In the wave theory, the validity of geometrical acoustics turns out to be the domain of short wavelengths. The validity of the geometrical acoustics is therefore limited by a lower frequency (the so-called Schroeder's frequency in room acoustics). To be complete, we must also mention that it exists also a upper frequency imposed by the limits of continuum mechanics.

The motivation of statistical energy analysis was exactly the converse. The fundamental laws of vibroacoustics were known for several decades at the beginning of statistical energy analysis. However, the former contributors raised the question of the applicability of classical sound and

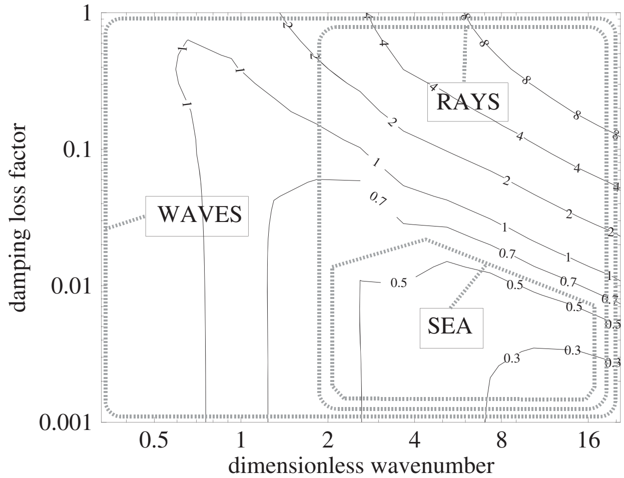

Figure 2. Domains of validity statistical energy analysis (SEA, bottom right-hand corner), geometrical acoustics (RAYS, right-side) and the wave theory (WAVES, entire plane) in the κ, η-plane of a thin rectangular plate in vibration. The isovalues give the relative standard deviation σ of vibrational energy.

vibration theory when the number of modes is very large, so large that a practical resolution of classical equations is unthinkable. The question was not that of correctness of the wave theory but rather its usefulness and relevance.

The construction of the theory was achieved through the discovery of the coupling power proportionality which enabled to set an analogy with Clausius principle in thermodynamics. This law highlights that under certain conditions, a thermalization of acoustical waves is possible. This law allowed the development of a thermodynamics of acoustical waves. Technically, a complex system is divided into several subsystems in diffuse field state. For all of them, the energy balance states the equality between the power delivered by external forces and the power lost by internal dissipation or exchanged with adjacent subsystems. This leads to a set of linear equations on subsystems energies. The solution gives the macroscopic repartition of vibrational energy among subsystems without solving the wave equation. This is a true thermodynamical approach of vibroacoustics based on the first principle.

The validity domain of statistical energy analysis is now well known [18]. The most important assumption is the diffuse field state that must be reached by all subsystems. For the sake of simplicity, we confine the discussion to a single subsystem. Let us examine the situation with only two controlled parameters chosen as the dimensionless wavenumber κ (number of wavelengths per mean-free-path) and the damping loss factor η. In figure 2 is plotted the relative standard deviation σ of the map of vibrational energy in a rectangular plate (the frequency bandwidth is an octave). $\sigma = 1$ means 100% of spatial variation about the mean value. A small value of σ means a homogeneous field of energy inside the plate that may be interpreted as the diffuse field state. The validity domain of statistical energy density is therefore the region of small σ. This is the region in the bottom right-hand corner of figure 2. It is characterized by large values of κ (high frequency or large number of modes) and small damping η. This last condition may be understood by remarking that when absorption is high, rays cannot propagate on long distances. They are reflected few times on boundaries before because they vanish rapidly and therefore rays cannot mix efficiently the energy. Similarly, the region of geometrical acoustics is delimited by its lowest frequency limit. This is the domain on the right-hand side of the vertical line in figure 2. Finally, the domain of Love's equation of thin plate is the entire κ, η-plane. This is the reference theory in this example.

The previous discussion on the validity domains highlights that it exists a strict hierarchy in the three theories. The largest one is the wave theory. It embodies geometrical acoustics as an approximation at high frequencies. Geometrical acoustics may therefore be seen as a theory derived from the wave theory. However, it turns out that statistical energy analysis is also a subtheory of geometrical acoustics [19]. Statistical energy analysis is an approximation of geometrical acoustics when the damping is low. We therefore observe the following hierarchy: wave → rays → statistical energy analysis.

The originality of the historical construction of statistical energy analysis is that formerly it was not strictly necessary to set a further theory. Waves and rays both cover a large validity domain sufficient for all practical purposes of engineers. However, the discovery of the coupling power proportionality opened the door to a new field in sound and vibration, the domain of disordered vibrations. When a vibrating system is sufficiently disorganized, the state of diffuse energy emerges and leads to a simpler behaviour. The advantage of a special theory well suited to this domain is simplicity but not generality. The idea is to deliberately construct a theory as a special case of a larger theory, just for convenience. This is rather unusual in the history of physical sciences.

However, there exist other examples in physics. For instance, the research of a closed theory of turbulence in fluid mechanics falls within the same idea. Basically, the fundamental of fluid mechanics is known (Navier–Stoke equation). However, the domain of high Reynold numbers exhibits a particular behaviour of flow that is highly disordered. Although the behaviour of turbulent flow is embodied in Navier–Stoke equation, it is obviously of interest to set a statistical theory of turbulence. This theory is not yet fully established.

6. Conclusion

We have shown in this article that the construction of statistical energy analysis, a sort of thermodynamics of sound and vibration, follows from a motivation rather from atypical in the history of physical sciences. The hierarchy of the three theories wave theory, geometrical acoustics and statistical energy analysis establishes a logical chain from the most general to the most particular theory. Each theory of this chain is a special case of the upper theory. The historical chronology in physics generally goes from the most particular to the most general theory. The reductionist programme in philosophy of science aims to find the logical link between existing theories and to construct a single framework theory to interpret them. However, the recent construction of statistical energy analysis represents a remarkable example of the reverse movement. Statistical energy analysis has been deliberately elaborated as a special case of a *previously existing* upper theory. There were therefore no theoretical requirement to set a new theory, but just a practical convenience to have at one's disposal a new simple theory that applies within a restricted but useful domain. This idea, which consists of increasing the number of special theories, establishes a new paradigm which rather falls within emergentism in philosophy of science.

Authors' contributions. A.L.B. conceived the mathematical model of entropy and its interpretation, wrote the paper and gave final approval for publication.
Competing interests. I declare I have no competing interests.
Funding. This work was supported by the CNRS and Labex CeLyA of Université de Lyon, operated by the French National Research Agency (ANR-10-LABX-0060/ANR-11-IDEX-0007). These supports are greatly appreciated.
Acknowledgements. The author acknowledges the referees for their positive comments.

References

1. Sabine WC. 1922 *Collected papers on acoustics*. Cambridge, MA: Harvard University Press.
2. Lyon RH, Maidanik G. 1962 Power flow between linearly coupled oscillators. *J. Acoust. Soc. Am.* **34**, 623–639. (doi:10.1121/1.1918177)
3. Newland DE. 1966 Calculation of power flow between coupled oscillators. *J. Sound Vib.* **3**, 262–276. (doi:10.1016/0022-460X(66)90095-2)

4. Scharton TD, Lyon RH. 1968 Power flow and energy sharing in random vibration. *J. Acoust. Soc. Am.* **43**, 1332–1343. (doi:10.1121/1.1910990)
5. Newland DE. 1968 Power flow between a class of coupled oscillators. *J. Acoust. Soc. Am.* **43**, 553–559. (doi:10.1121/1.1910865)
6. Carcaterra A. 2002 An entropy formulation for the analysis of energy flow between mechanical resonators. *Mech. Syst. Signal Process.* **16**, 905–920. (doi:10.1006/mssp.2002.1486)
7. Carcaterra A. 2005 Ensemble energy average and energy flow relationships for nonstationary vibrating systems. *J. Sound Vib.* **288**, 751–790. (doi:10.1016/j.jsv.2005.07.015)
8. Le Bot A. 2009 Entropy in statistical energy analysis. *J. Acoust. Soc. Am.* **125**, 1473–1478. (doi:10.1121/1.3075613)
9. Le Bot A, Carcaterra A, Mazuyer D. 2010 Statistical vibracoustics and entropy concept. *Entropy* **12**, 2418–2435. (doi:10.3390/e12122418)
10. Le Bot A. 2011 Statistical energy analysis and the second principle of thermodynamics. In *IUTAM Symposium on the Vibration Analysis of Structures with Uncertainties*, pp. 129–139. London, UK: Springer. (doi:10.1007/978-94-007-0289-9)
11. Tufano D, Sotoudeh Z. 2017 Exploring the entropy concept for coupled oscillators. *J. Int. Eng. Sci.* **112**, 18–31. (doi:10.1016/j.ijengsci.2016.12.003)
12. Tufano D, Sotoudeh Z 2016 Exploring entropy for continuous vibrating systems. In *57th AIAA/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conf., San Diego, CA, 4–8 January*.
13. Carcaterra A, Akay A. 2004 Transient energy exchange between a primary structure and a set of oscillators: return time and apparent damping. *J. Acoust. Soc. Am.* **115**, 683–696. (doi:10.1121/1.1642619)
14. Carcaterra A, Akay A. 2007 Theoretical foundations of apparent-damping phenomena and nearly irreversible energy exchange in linear conservative systems. *J. Acoust. Soc. Am.* **121**, 1971–1982. (doi:10.1121/1.2697030)
15. Le Bot A. 2015 *Foundation of statistical energy analysis in vibroacoustics*. Oxford, UK: Oxford University Press.
16. Mace BR, Ji L. 2007 The statistical energy analysis of coupled sets of oscillators. *Proc. R. Soc. A* **463**, 1359–1377. (doi:10.1098/rspa.2007.1824)
17. Lafont T, Totaro N, Le Bot A. 2013 Review of statistical energy analysis hypotheses in vibroacoustics. *Proc. R. Soc. A* **470**, 20130515. (doi:10.1098/rspa.2013.0515)
18. Le Bot A, Cotoni V. 2010 Validity diagrams of statistical energy analysis. *J. Sound Vib.* **329**, 221–235. (doi:10.1016/j.jsv.2009.09.008)
19. Le Bot A. 2007 Derivation of statistical energy analysis from radiative exchanges. *J. Sound Vib.* **300**, 763–779. (doi:10.1016/j.jsv.2006.08.033)