

## STATISTICAL ENERGY ANALYSIS AND DIFFUSE FIELD

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Statistical energy analysis (SEA) is a statistical theory of sound and vibration based on an analogy with thermodynamics. The main relationship of statistical energy analysis, the so-called coupling power proportionality, is intimately linked with the establishment of a diffuse vibration field in subsystems. In this study, we explore the conditions under which a diffuse field is enforced. We show that when the subsystem is excited by a point force, a low damping, a high frequency but also a ergodic billiard geometry are required conditions. Then, the energy exchange between two weakly coupled subsystems is proportional to the difference of vibrational energies. But when the field is not diffuse, the exchange of energy does not generally follow this proportionality. Numerical simulations are provided to support the discussion.

Keywords: sound, vibration, statistical energy analysis, diffuse field, ergodic billiard

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### 1. Introduction

Statistical energy analysis [1, 2, 3, 4, 5, 6] is a statistical theory of sound and vibration in the same way it exists a statistical mechanics, statistical physics, statistical electromagnetism. The basic concept is that of diffuse field. A vibrational field, either structural or acoustical, is diffuse if it is random, homogeneous and isotropic [7]. Nowadays, statistical energy analysis represents the most popular method in high frequency vibroacoustics although extensions and/or alternative methods have been developed [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

Identifying the exact assumptions of statistical energy analysis is of primary importance for both, theoretical and practical points of view. The success in applications is conditioned by the knowledge of

this list. But if the main assumptions are well identified, it seems that the exact list of assumptions may slightly differ for the authors [20, 21, 22, 23, 24].

In this study, we explore the status of the assumption of ergodicity of subsystems in statistical energy analysis [25]. For this purpose, we shall systematically compare the prediction of statistical energy analysis with a reference calculation obtained by a direct numerical simulation. For each test case, we shall verify whether or not the exchanged power is proportional to the difference of modal energies, as claimed in statistical energy analysis.

## 2. Statistical energy analysis

To test whether or not, the geometry of subsystems must be ergodic to ensure the validity of statistical energy analysis, we consider a simple system composed of two coupled subsystems. The subsystems are thin plates in flexural vibrations coupled by a spring. The plates are excited by a stationary random moment whose spectrum is flat and confined within the bandwidth  $\Delta\omega$ .

The modal density of plate  $i$  defined as the number of modes per unit circular frequency  $\omega$  (rad/s) is

$$n_i = \frac{A_i}{4\pi} \sqrt{\frac{m_i}{D_i}} \quad (1)$$

where  $m_i$  is the mass per unit area,  $D_i$  the bending stiffness, and  $A_i$  the plate area. The number of resonant modes in the frequency bandwidth  $\Delta\omega$  is  $N_i = n_i \Delta\omega$ .

We denote by  $E_i$  the expectation of vibrational energy in subsystem  $i$ . Expectation must be understood in the meaning of probability theory since the excitation is random. This is the mean value for several realizations of the random process associated to excitation moment. By virtue of stationarity,  $E_i$  does not depend on time. We also denote by  $P_{ij}$  the expectation of vibrational power exchanged between subsystems  $i$  and  $j$ . Then, the main result of statistical energy analysis states the power being exchanged between subsystems  $i$  and  $j$  is proportional to the difference of modal energies

$$P_{ij} = \beta_{ij} \left( \frac{E_i}{n_i} - \frac{E_j}{n_j} \right) \quad (2)$$

This is the coupling power proportionality. The conductivity factor  $\beta_{ij}$  verifies reciprocity  $\beta_{ij} = \beta_{ji}$ . For two plates coupled by a spring, its value is given by [22]

$$\beta_{ij} = \frac{K^2}{32\pi\omega^2 \sqrt{m_i D_i m_j D_j}} \quad (3)$$

where  $K$  is the stiffness of the coupling spring.

## 3. Ergodic billiard

We selected two different geometries for plates whose properties of ray dynamics are well-known. The stadium geometry is a typical example of ergodic billiard [26, 27]. This imposes that any ray explores during its travel the entire phase space *i.e.* passes through the vicinity of all points and all directions. An example of ray path is given in Fig. 1, left. It can be seen that a single ray spread its energy uniformly and isotropically over the subsystem. This is exactly what is required to get a diffuse field. This illustrates the fact that an ergodic billiard ensures a diffuse field state even if the excitation is a single point and has a high directivity.

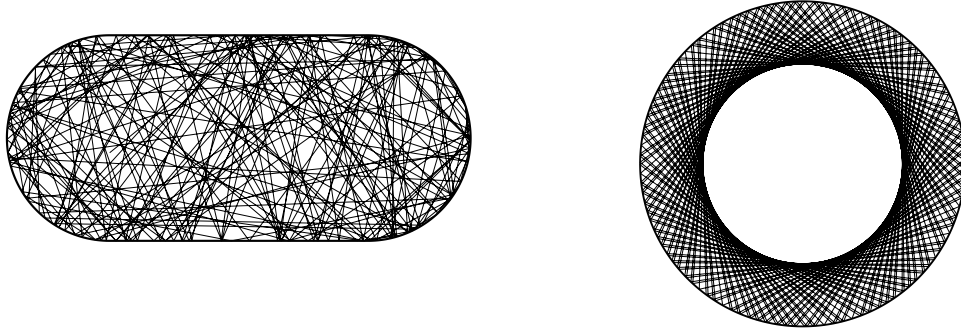


Figure 1: Ray dynamics in various billiards. Left, ray tracing in a stadium. Right, ray tracing in a circle.

The second example is a circular geometry. The conservation of incident angle on boundary is a feature particular to circle. An example of ray path is shown in Fig. 1, right. It can be seen that the ray turns inside the circle but never explores the central zone of the circle. This explains why a circle is not an ergodic billiard and therefore never gives a diffuse field.

#### 4. Parameters of simulation

The parameters of the simulations are the following. The plates are all made of steel with Young's modulus  $E_0 = 210$  GPa, density  $\rho = 7800$  kg.m<sup>-3</sup> and Poisson's ratio  $\nu = 0.3$ . The plates have a damping loss factor  $\eta = 0.001$ . Plate 1 has thickness 2 mm and mass per unit area  $m_1 = 15.6$  kg/m<sup>2</sup> while plate 2 has thickness 2.5 mm and mass per unit area  $m_2 = 19.5$  kg/m<sup>2</sup>. Plate 1 is excited by a moment, or torque, simulated by two closed forces. The forces are out-of-phase but random. The frequency band of excitation is an octave centred on  $\omega = 2\pi \times 4000$  rad.s<sup>-1</sup>.

The size of plates is the following. The width of stadium is 0.7 m and the length is 1.6 m so that its area is  $A_1 = A_2 = 1$  m<sup>2</sup>. The circle has diameter 1.13 m and has area  $A_1 = A_2 = 1$  m<sup>2</sup>. These dimensions have been chosen to ensure a same number of modes for stadium and circle in the octave band 4000 Hz (see Tab. 1). The damping  $\eta$  is light so that the attenuation of wave per mean-free path,  $\eta\omega l/c$  where  $c$  is the group speed and  $l$  the mean-free path, is small. A small attenuation ensures that rays reflect a large number of times before to vanish [28]. The coupling spring stiffness is  $K = 1 \times 10^5$  N/m. The connection strength  $\kappa = K/(m_1 A_1 m_2 A_2 \omega^4)^{1/2}$  is small so that the coupling is weak [29, 30, 31].

Table 1: Parameters of plates.

Plate	# Modes	Attenuation	Modal overlap	Connection strength
Stadium 1	450	0.036	0.64	$10^{-5}$
Stadium 2	360	0.032	0.51	-
Circle 1	450	0.040	0.64	$10^{-5}$
Circle 2	360	0.035	0.51	-

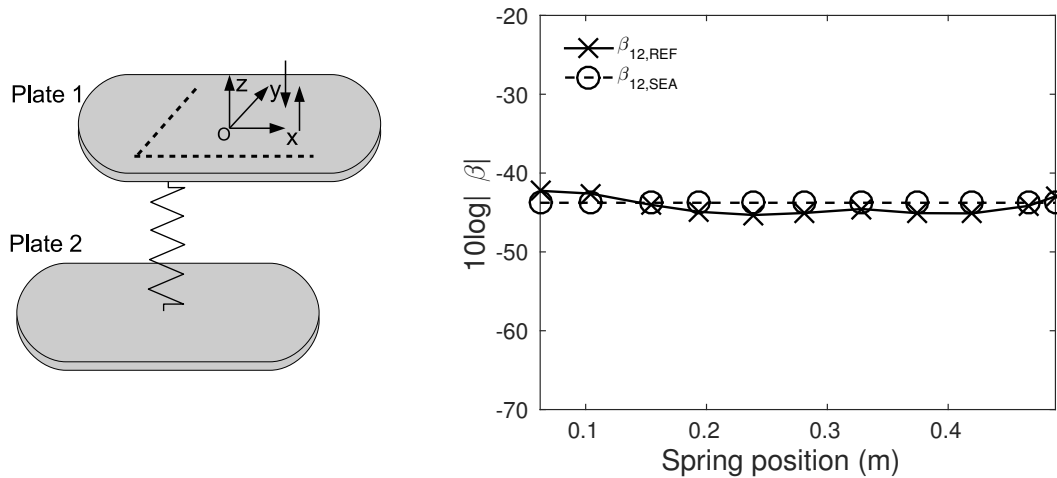


Figure 2: Comparison of energy transfer predicted by statistical energy analysis and reference calculation for two coupled stadium plates excited by a torque on plate 1. Factor  $\beta$  given by Eq. (3) and reference calculation versus position of coupling spring (dashed line in left).

## 5. Results and comments

In Fig. 2 is shown the ratio  $\beta$  of exchanged power and difference of modal energies calculated respectively from Eq. (3) and by the reference calculation in the case of two coupled stadiums. It is observed that a fine agreement is found between statistical energy analysis and reference for any position of the coupling spring. This agreement is remarkable since the excitation is a unique torque applied. This type of excitation is severe for statistical energy analysis since it is highly localized and highly directive. We are far from the traditional rain-on-the-roof excitation. In this case, homogeneity and isotropy of vibrational field are guaranteed by ergodicity of stadium billiard, while a torque excitation provides neither homogeneity nor isotropy of source.

In Fig. 3 is shown the ratio  $\beta$  in same conditions but for two coupled circular plates. We observe a large disagreement between statistical energy analysis prediction and that of the reference calculation. The reason is that since the excitation is highly localized and directive, and that the subsystems are not ergodic, the vibrational field is never diffuse. The exchanged of energy between the two plates do not follow the thermal law of Eq. (2) since the thermal equilibrium is not reached in subsystems.

## 6. Conclusion

Diffuse field is an imperative assumption of statistical energy analysis. The requirement of a rain-on-the-roof excitation is a condition which ensures diffuseness of the field. But when the excitation is random but localized and/or highly directive, ergodicity of subsystem is an important condition to get a diffuse field. The counter-example of circular plates show that even if the frequency is high, the attenuation is low, and the coupling is weak, the coupling power proportionality may be violated when subsystems do not have ergodic geometries.

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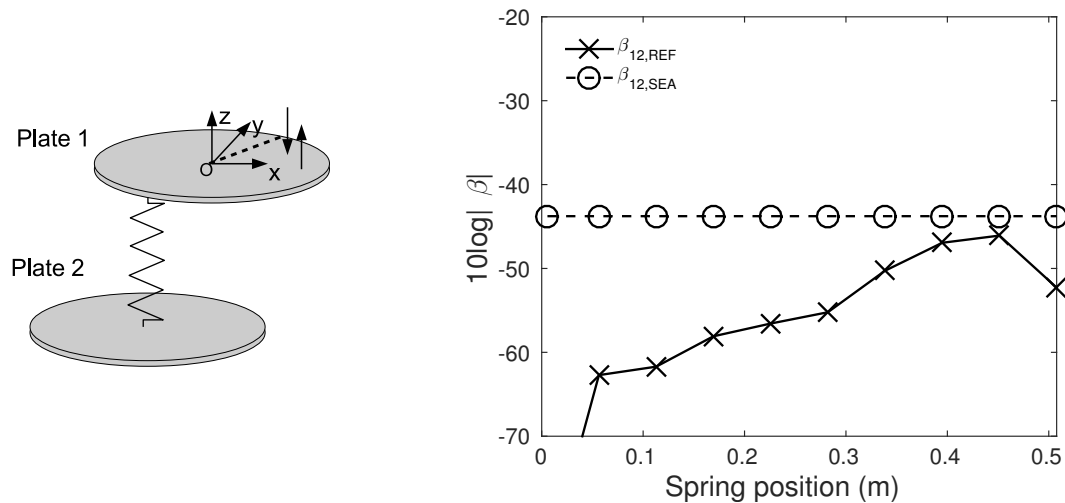


Figure 3: Comparison of energy transfer predicted by statistical energy analysis and reference calculation for two coupled circular plates excited by a torque on plate 1. Factor  $\beta$  given by Eq. (3) and reference calculation versus position of coupling spring (dashed line in left).

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